Vanishing cycles and singularities of meromorphic functions

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Abstract

We study vanishing cycles of meromorphic functions. This gives a new and unitary point of view, extending the study of the topology of holomorphic germs – as initiated by Milnor in the sixties – and of the global topology of polynomial functions, which has been advanced more recently. We define singularities along the poles with respect to a certain (weak) stratification and prove local and global bouquet structure in case of isolated singularities. In general, splitting of vanishing homology at singular points and global Picard-Lefschetz phenomena occur.

key words: vanishing cycles, singularities along the poles, topology of meromorphic functions

1 Introduction

Let $\mathcal{Z}$ denote a connected compact complex manifold. We consider a meromorphic function $f : \mathcal{Z} \dashrightarrow \mathbb{C}$, as defined in [GR]. This defines a holomorphic function on $\mathcal{Z} \setminus \mathrm{Pol}(f)$, where $\mathrm{Pol}(f)$ is a hypersurface (or void) in $\mathcal{Z}$, the set of poles. We denote by $\mathrm{Z}(f)$ the set of zeroes of f, a hypersurface in $\mathcal{Z}$. Locally at some point $x \in \mathcal{Z}$, the meromorphic function f has the form $f = p/q$, where p and q are germs of holomorphic functions.

The scope of this paper is to define and study vanishing cycles of meromorphic functions. We prove results in both local and global context. The local results continue some of Milnor's [Mi] on holomorphic functions while the global results extend some of those proved more recently on polynomial functions on affine spaces.

We set $X := \{(x,t) \in (\mathcal{Z} \setminus \mathrm{Pol}(f)) \times \mathbb{C} \mid f(x) - t = 0\}$ and we identify the fibres of the function $f : \mathcal{Z} \setminus \mathrm{Pol}(f) \to \mathbb{C}$ to the fibres of the projection $\pi : X \to \mathbb{C}$, $(x,t) \mapsto t$. Remark that X is isomorphic to $\mathcal{Z} \setminus \mathrm{Pol}(f)$, in particular it is nonsingular. Several aspects of our paper apply to singular $\mathcal{Z}$.

1.1 Definition We call *completed space* the global hypersurface $\mathbb{X}$ of $\mathcal{Z} \times \mathbb{P}^1$ defined locally by:

$$sp(z) - tq(z) = 0,$$

where $f_x = p/q$ and z is close enough to x. We denote by $\bar{\pi} : \mathbb{X} \to \mathbb{P}^1$ the proper projection which extends the projection $X \to \mathbb{C}$. The analytic closure of X in $\mathcal{Z} \times \mathbb{P}^1$ is just $\mathbb{X}$. The space $\mathbb{X}^{\mathrm{pol}} := \mathbb{X} \cap (\mathrm{Pol}(f) \times \mathbb{P}^1) \subset \mathbb{X}$ is a divisor of $\mathbb{X}$ and $X = \mathbb{X} \setminus \mathbb{X}^{\mathrm{pol}}$.

The same construction can be done in case of a meromorphic germ, by taking instead of the "big" space $\mathcal{Z}$, just a small ball neighbourhood. Throughout this paper we shall tacitly use this construction in case of germs. All our results are therefore valid in this setting too.

The first result one may observe for the fibration defined by a meromorphic function is that atypical values are finitely many, as in case of polynomial functions. By endowing $\mathbb{X}$ with a locally finite Whitney stratification $\mathcal{W}$ such that X is a stratum, and by using Verdier's technique [Ve], one may prove the following statement, which has its origin in Thom's paper [T]:

1.2 Proposition [Ve]
The stratified projection $\bar{\pi} : \mathbb{X} \to \mathbb{P}^1$ with respect to $\mathcal{W}$ is locally topologically trivial over $\mathbb{C} \setminus \Lambda_f$, for some finite set Λ_f. The restriction $\bar{\pi}_{|X} = \pi$ is a locally trivial C^∞-fibration over $\mathbb{C} \setminus \Lambda_f$. □

It follows that the meromorphic function f is C^∞ locally trivial over $\mathbb{C} \setminus \Lambda_f$. (This result works of course in case of a meromorphic germ.)

For any subset $A \subset \mathbb{P}^1$, we denote $\mathbb{X}_A := \bar{\pi}^{-1}(A)$, $F_A := \pi^{-1}(A) = f^{-1}(A)$ and the general fibre $F := F_t = X \cap \pi^{-1}(t) = f^{-1}(t)$, for some $t \notin \Lambda_f$. Let D be a small disc centered at $a \in \Lambda_f$, such that $D \cap \Lambda_f = \{a\}$.

1.3 Definition We call $H_*(X, F)$ the *(global) vanishing homology* of f and we call $H_*(F_D, F)$ the *(global) vanishing homology at $a \in \Lambda_f$* of f. The *(local) vanishing homology* of f at $\xi \in \mathbb{X}$ is $H_*(F_D \cap B, F \cap B)$, where B is a small ball at ξ within $\mathcal{Z} \times \mathbb{P}^1$ and the radius of D is very small in comparison to the radius of B.

The vanishing homology of meromorphic functions is a natural extension of vanishing homology of local holomorphic functions, for which the total space is contractible. In case of our total space X, resp. $F_D \cap B$, the general fibre inherits part of the homology from X, resp. $F_D \cap B$, which part will not vanish (if the total space is not contractible). The pair (X, F) reflects the embedded nature of F into X.

The consideration of meromorphic instead of holomorphic leads to a new type of singularities, those along the poles $\mathrm{Pol}(f)$ of f. To treat such singularities we use a *partial Thom stratification* $\mathcal{G}$ along $\mathrm{Pol}(f)$ (introduced in [Ti-1]) which is less restrictive (and easier to use) than a Whitney stratification. In case of an isolated point of the singular locus $\mathrm{Sing}_{\mathcal{G}} f$, situated on $\mathrm{Pol}(f)$, we attach to it a *polar Milnor number*. If f has only isolated singularities along $\mathrm{Pol}(f)$, then this number turns out to be the number of vanishing cycles which are "concentrated" at this singularity (Proposition 3.7).

In this case, not only that the vanishing homology is concentrated in dimension n, but the space X/F itself has the homotopy type of a bouquet of spheres $\bigvee S^n$ (Theorem 3.8). The proof needs new technical ingredients, since the notion of isolated singularity that we use (i.e. with respect to a partial Thom stratification) is more general than what one usually uses. Proposition 4.2 and Lemma 4.3 are crucial in this respect. It also appears that the polar Milnor number λ_ξ at $\xi \in \mathbb{X}^{\mathrm{pol}} \setminus \mathbb{X}_\infty$ coincides with the jump at ξ of the local Milnor numbers within the pencil of hypersurfaces (whenever this local Milnor number

is defined), see Theorem 5.1. Another application is the following equisingularity result along $\mathrm{Pol}(f)$: If f has isolated $\mathcal{G}$-singularities at ξ, resp. at $a \in \mathbb{C}$, then f is C^∞-trivial along $\mathrm{Pol}(f)$ at ξ, resp. at $a \in \mathbb{C}$, if and only if $\lambda_\xi = 0$, resp. $\lambda_a = 0$ (see Theorem 4.12). In dimension 2, f has isolated $\mathcal{G}$-singularities (for a natural partial Thom stratification $\mathcal{G}$) if and only if its fibres are reduced.

Significant particular cases and examples are treated in §5. Les us give one here. Start with the nonsingular hypersurface $\mathcal{Z} \subset \mathbb{P}^3$, given by $h = x^2 + y^2 + zw = 0$ and consider the meromorphic function $f = y/x$. Then f has two critical points on $\mathcal{Z} \setminus \mathrm{Pol}(f)$ and there are no jumps along $\mathbb{X}^{\mathrm{pol}}$ (thus no vanishing cycles along $\mathbb{X}^{\mathrm{pol}}$). There is a total number of $\mu = 2$ vanishing cycles (see Definition 4.5). A different situation may occur if the pencil defined by f has its axis $\{x = y = 0\}$ tangent to $h = 0$; this happens e.g. for $h = x^2 + z^2 + yw$. By computations we get $\mu = 0$, $\lambda = 1$ (jump $A_0 \to A_1$, at $t = 0$). The general fibre F is contractible, the special fibre F_0 is $\mathbb{C} \sqcup \mathbb{C}$ and X is homotopy equivalent to S^2. All the connected components of fibres are contractible, but the global vanishing homology is generated by a relative 2-cycle. An example of a non-linear pencil is considered in §5.

We prove in general the decomposition of global vanishing homology into the sum of vanishing homologies at the atypical fibres (Proposition 2.1), with localisation in the case of $\mathcal{G}$-isolated singularities (Proposition 3.7), and reveal the existence of a global Picard-Lefschetz phenomenon for the monodromy (Proposition 2.2).

From the point of view of this paper, investigating the topology of a polynomial function $g : \mathbb{C}^n \to \mathbb{C}$ at infinity amounts to consider the rational function $\tilde{g}/x_0^d$, where $d = \deg g$, $\tilde{g}$ is the homogenized of g and x_0 is the variable at infinity in the total space $\mathcal{Z} = \mathbb{P}^n$.

Our interest for meromorphic functions came, on the one hand from the recent study of the topology of polynomial functions at infinity and of one parameter families of non compact hypersurfaces, in particular [ST-1], [Ti-1,3], [ST-2], and on the other hand from the recent classification of the simple germs of meromorphic functions under certain equivalence relations ($\mathcal{R}$ and $\mathcal{R}^+$), by Arnold [Ar].

2 Monodromy fibration and global Picard-Lefschetz phenomenon

We consider a big closed disc $\hat{D} \subset \mathbb{C}$ which contains all the atypical values Λ_f of f. Let $D_i \subset \hat{D}$ be a small enough closed disc at $a_i \in \Lambda_f$, such that $D_i \cap \Lambda_f = \{a_i\}$. Take a point s on the boundary of $\hat{D}$ and, for each i, a path $\gamma_i \subset \hat{D}$ from s to some fixed point $s_i \in \partial D_i$, with the usual conditions: the path γ_i has no self intersections and does not intersect any other path γ_j, except at the point s. By Proposition 1.2, the fibration $\pi : X \setminus \pi^{-1}(\Lambda_f) \to \mathbb{C} \setminus \Lambda_f$ is locally trivial, hence we may use excision in the pair $(F_{\hat{D}}, F_s)$ and get an isomorphism (induced by the inclusion of pairs):

$$(1) \qquad H_*(\pi^{-1}(\cup_{a_j \in \Lambda_f} \gamma_j \cup D_j), F_s) \xrightarrow{\simeq} H_*(X, F_s)$$

The following statement traces back to our excision result [ST-1, §3]:

2.1 Proposition *Let $f : \mathcal{Z} \dashrightarrow \mathbb{C}$ be a meromorphic function. Then*

(a) *The vanishing homology of f is the direct sum of the vanishing homologies at a_i, for a_i running in Λ_f:*
$$H_*(X, F) = \oplus_{a_i \in \Lambda_f} H_*(F_{D_i}, F).$$

(b) *The long exact sequence of the triple (X, F_{D_i}, F) decomposes into short exact sequences:*
$$0 \to H_*(F_{D_i}, F) \to H_*(X, F) \to H_*(X, F_{D_i}) \to 0.$$

(c) *There is a natural identification $H_*(X, F_{D_i}) = \oplus_{a_j \in \Lambda_f, j \neq i} H_*(F_{D_j}, F)$. The short exact sequence of* (b) *is split and naturally identifies to the following exact sequence:*

$$0 \to H_*(F_{D_i}, F) \to H_*(X, F) \to \oplus_{a_j \in \Lambda_f, j \neq i} H_*(F_{D_j}, F) \to 0. \tag{2}$$

Proof We may further excise in the relative homology $H_*(\pi^{-1}(\cup_{a_j \in \Lambda_f} \gamma_j \cup D_j), F_s)$ from (1) and get an isomorphism:

$$\oplus_{a_i \in \Lambda_f} H_*(F_{D_i}, F_{s_i}) \to H_*(X, F_s), \tag{3}$$

which is induced by inclusion of pairs of spaces.

This also shows that each inclusion $(F_{D_i}, F_{s_i}) \subset (X, F_{s_i})$ induces an injection in homology $H_*(F_{D_i}, F_{s_i}) \hookrightarrow H_*(X, F_{s_i})$. All the points (a), (b), (c) follow from this. Note that (c) also follows by excision. $\square$

We next consider the monodromy h_i around an atypical value $a_i \in \Lambda_f$. This is induced by a counterclockwise loop around the small circle ∂D_i. The monodromy acts on the pair (X, F) and we denote its action in homology by T_i.

The following sequence of maps:

$$H_{q+1}(X, F) \xrightarrow{\partial} H_q(F) \xrightarrow{w} H_{q+1}(F_{\partial D_i}, F) \xrightarrow{i_*} H_{q+1}(X, F), \tag{4}$$

where w denotes the Wang map (which is an isomorphism, by the Künneth formula), gives, by composition, the map:

$$T_i - \mathrm{id} : H_{q+1}(X, F) \to H_{q+1}(X, F).$$

This overlaps the first two maps in the following sequence, which defines $T_i - \mathrm{id} : H_q(F) \to H_q(F)$:

$$H_q(F) \xrightarrow{w} H_{q+1}(F_{\partial D_i}, F) \xrightarrow{i_*} H_{q+1}(X, F) \xrightarrow{\partial} H_q(F).$$

The last arrow in the sequence (4) fits in the commutative diagram:

$$\begin{tikzcd} H_{q+1}(F_{\partial D_i}, F) \arrow[rr, "i_*"] \arrow[dr] & & H_{q+1}(X, F) \\ & H_{q+1}(F_{D_i}, F) \arrow[ur] & \end{tikzcd}$$

where all three arrows are induced by inclusion.

It follows that the submodule of "anti-invariant cycles" $\mathcal{I}_*(T_i) := \mathrm{Im}(T_i - \mathrm{id} : H_*(X,F) \to H_*(X,F))$ is contained in the direct summand $H_*(F_{D_i}, F)$ of $H_*(X,F))$. Hence the following *global Picard-Lefschetz phenomenon* occurs: the action of the monodromy T_i on a vanishing cycle $\omega \in H_*(X,F)$ changes ω by adding to it only contributions from the cycles vanishing at a_i.

2.2 Proposition *Identify $H_*(X,F)$ to $\oplus_{a_i\in\Lambda_f} H_*(F_{D_i}, F)$ by the isomorphism* (3)*. Then, for $\omega \in H_*(X,F)$, we have:*

$$T_i(\omega) = \omega + \psi_i(\omega),$$

for some $\psi_i(\omega) \in H_*(F_{D_i}, F)$. □

Let $\mathcal{I}_*$ denote the submodule generated by all $\mathcal{I}_*(T_i)$, for $a_i \in \Lambda_f$. Then:

$$\mathcal{I}_* = \oplus_{a_i\in\Lambda_f} \mathcal{I}_*(T_i). \tag{5}$$

2.3 Notes (a) Specializing to a homologically trivial total space X, the natural ∂-map $H_*(X,F) \to \tilde{H}_{*-1}(F,\mathbb{Z})$ becomes an isomorphism and we get for instance:

$$H_*(F) = \oplus_{a_i\in\Lambda_f} H_*(F_{D_i}, F).$$

(b) The above results "dualize" easily from homology to cohomology. One obtaines in this way statements about invariant cocycles $\mathrm{Ker}(T^i - \mathrm{id} : H^*(X,F) \to H^*(X,F))$.

(c) A special case of point (a) is that of a polynomial function $g : \mathbb{C}^n \to \mathbb{C}$, for which $X \simeq \mathbb{C}^n$. In this case, results on invariant cocycles were obtained in [ACD, Th. 1 and 2, Cor. 1], under strong restrictions (general fibre has concentrated homology in highest dimension, the atypical fibre F_{s_i} has only isolated singularities) and by more involved proofs. In a recent manuscript [NN], these results are proved in a natural way and in whole generality for a polynomial function. They can be proved in even more generality, i.e. in our setting, as explained at point (b). The reader who wants to fill in the details of this program may refer to [NN] as a guideline.

(d) Gusein-Zade, Luengo and Melle recently focused on finding formulas for the zeta-function of the monodromy, [GLM-1,2].

3 Singularities of f along the poles

A crucial problem in investigating the topology of the fibres of f is how to detect and to control the change of topology. In the context of meromorphic germs, one first has to define a local fibration of p, which is done bellow.

3.1 Definition Let $f_x : (\mathcal{Z}, x) \dashrightarrow \mathbb{C}$ be a germ of a meromorphic function. One associates to it the germ $\bar{\pi} : (\mathbb{X}, (x, a)) \to \mathbb{P}^1$, for $a \in \mathbb{P}^1$ and, by restriction to $X = \mathbb{X} \setminus \mathbb{X}^{\rm pol}$, the corresponding "generalised germ" $\pi : (X, (x, a)) \to \mathbb{C}$, in the sense that we allow the point (x, a) be in the closure of the set X.

In case $x \notin {\rm Pol}(f) \cap {\rm Z}(f)$, where ${\rm Z}(f)$ is the divisor of zeros of f, this germ is uniquely determined, by the determination of the point $a = [p(x) : q(x)]$. If $x \in {\rm Pol}(f) \cap {\rm Z}(f)$, one has a *one-parameter family of germs*, indexed over $a \in \mathbb{P}^1$.

Remark that ${\rm Sing}\, \mathbb{X} \subset \mathbb{X}^{\rm pol} \cup \mathbb{X}_\infty$. Take a Whitney stratification $\mathcal{W}$ of $\mathbb{X}$ which has X as open stratum. For a ball $B_\varepsilon(x, a) \subset \mathcal{Z} \times \mathbb{P}^1$ centered at (x, a), for all small enough radii ε, the sphere $S_\varepsilon = \partial \bar{B}_\varepsilon(x, a)$ intersects transversally all the finitely many strata in the neighbourhood of (x, a). This fact implies, according to [Le-2], that there is a locally trivial fibration $\bar{\pi} : \mathbb{X}_{D^*} \cap B_\varepsilon(x, a) \to D^*$ which restricts to a locally trivial fibration on the complement of $\mathbb{X}^{\rm pol}$, namely:

$$\pi : F_{D^*} \cap B_\varepsilon(x, a) \to D^*. \tag{6}$$

From Proposition 1.2 it follows that, since $\bar{\pi}$ is stratified-transversal to $\mathbb{X}$ over $\mathbb{C} \setminus \Lambda_f$, the fibration (6) is trivial and moreover $\pi : F_D \cap B_\varepsilon(x, a) \to D$ is a trivial fibration, for all but a finite number of germs in the family along $\{x\} \times \mathbb{P}^1$ considered above.

3.2 Definition We call the locally trivial fibration (6) the *Milnor fibration* of the meromorphic function germ f_x at the point $(x, a) \in \mathbb{X}$.

In [GLM-1], Gusein-Zade, Luengo and Melle use a different definition for two special points (called "zero" and "infinite" Milnor fibrations in *loc cit*), which they later extend in [GLM-2]. In fact their definition is equivalent to ours.

We put on $\mathbb{X}$ a partial Thom stratification, following [Ti-1, §3.]. Suppose that $\mathbb{X}$ is endowed with a complex stratification $\mathcal{G} = \{\mathcal{G}_\alpha\}_{\alpha \in S}$ such that $\mathbb{X}^{\rm pol}$ is a union of strata. If $\mathcal{G}_\alpha \cap \overline{\mathcal{G}_\beta} \neq \emptyset$ then, by definition, $\mathcal{G}_\alpha \subset \overline{\mathcal{G}_\beta}$ and in this case we write $\mathcal{G}_\alpha < \mathcal{G}_\beta$.

Let $(x, a) \in \mathbb{X}^{\rm pol}$ and let $f_x = p/q$. Locally, $\mathbb{X}^{\rm pol}$ is defined by $q = 0$ but q is defined only up to a unit. We consider the *Thom* $({\rm a}_q)$ *regularity condition* at (x, a), see e.g. [GWPL, ch. I] for the definition. In terms of the *relative conormal* (see [Te], [HMS] for a definition), the condition $({\rm a}_q)$ at $\xi := (x, a)$ for the strata $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ translates to the inclusion: $T^*_{\mathcal{G}_\alpha} \supset (T^*_{g|\overline{\mathcal{G}_\beta}})_\xi$. It is known that this condition is independent on q, up to multiplication by a unit, see e.g. [Ti-1, Prop. 3.2]. We therefore may and shall refer to this as *Thom regularity condition relative to* $\mathbb{X}^{\rm pol}$*, at* (x, a).

3.3 Definition We say that $\mathcal{G}$ is a $\partial\tau$-stratification (*partial Thom stratification*) relative to $\mathbb{X}^{\rm pol}$ if the following condition is satisfied at any point $\xi \in \mathbb{X}$:

(*) any two strata $\mathcal{G}_\alpha < \mathcal{G}_\beta$ with $\xi \in \mathcal{G}_\alpha \subset \mathbb{X}^{\rm pol}$ and $\mathcal{G}_\beta \subset \mathbb{X} \setminus \mathbb{X}^\infty$ satisfy the Thom regularity condition relative to $\mathbb{X}^{\rm pol}$, at ξ.

Our Whitney stratification $\mathcal{W}$ of $\mathbb{X}$ is an example of $\partial\tau$-stratification relative to $\mathbb{X}^{\rm pol}$. This follows from Briançon, Maisonobe and Merle's result [BMM, Théorème 4.2.1], also

proven by Tibăr, using a different method [Ti-1, Theorem 3.9]. Nevertheless the $\partial\tau$-stratifications are less demanding than Whitney stratifications and than Thom stratifications. One can contruct a canonical (minimal) $\partial\tau$-stratification relative to $\mathbb{X}^{\mathrm{pol}}$ by following the construction in [Ti-1, §3].

3.4 Definition Let $\mathbb{X}_\infty := \bar{\pi}^{-1}([0:1])$. We define the singular locus of f with respect to $\mathcal{G}$, where $\mathcal{G}$ is some $\partial\tau$-stratification relative to $\mathbb{X}^{\mathrm{pol}}$, as the closed subset:

$$\mathrm{Sing}_{\mathcal{G}} f := (\mathbb{X} \setminus \mathbb{X}_\infty) \cap \cup_{\mathcal{G}_\alpha \in \mathcal{G}} \mathrm{closure}(\mathrm{Sing}\, \bar{\pi}_{|\mathcal{G}_\alpha}).$$

We say that f has *isolated singularities* with respect to $\mathcal{G}$ if $\dim \mathrm{Sing}_{\mathcal{G}} f \leq 0$. We say that f has isolated singularities at $a \in \mathbb{C}$ (or at the fibre $\mathbb{X}_a$) if $\dim \mathbb{X}_a \cap \mathrm{Sing}_{\mathcal{G}} f \leq 0$.

The space X is nonsingular and consists of one stratum. The space $X \cap \mathrm{Sing}_{\mathcal{G}} f$ of $\mathcal{G}$-singularities on X is just (isomorphic to) the usual space of singularities $\mathrm{Sing}\, f \subset \mathcal{Z} \setminus \mathrm{Pol}(f)$. The new type of singularities $\mathbb{X}^{\mathrm{pol}} \cap \mathrm{Sing}_{\mathcal{G}} f$ are those along the poles of f.

Isolated singularities are convenient to work with since in this case one may localize the variation of topology of fibres.

3.5 Definition We say that the variation of topology of the fibres of f at some fibre F_a is *localizable* if there exists a finite set $\{a_1, \ldots, a_k\} \in \mathbb{X}_a$ such that, for small enough balls $B_{\varepsilon,i} \subset \mathcal{Z} \times \mathbb{C}$, $i \in \{1, \ldots, k\}$, and small enough disc $D_a \subset \mathbb{C}$, the restriction $\pi : (X \setminus \cup_{i=1}^k B_{\varepsilon,i}) \cap F_{D_a} \to F_{D_a}$ is a trivial fibration.

The second author proved a general localisation theorem, which can be applied to our holomorphic function $\pi : X \to \mathbb{C}$. We state it in the form adapted to our use.

3.6 Proposition (see [Ti-1, Theorem 4.3])
Let f have isolated singularities with respect to $\mathcal{G}$ at $a \in \mathbb{C}$. Then the variation of topology of the fibres of f at F_a is localisable at those points. $\square$

The localization result implies that the vanishing cycles are concentrated at the isolated singularities, as follows:

3.7 Proposition *Let f have isolated singularities with respect to $\mathcal{G}$ at $a \in \mathbb{C}$ and let $\mathbb{X}_a \cap \mathrm{Sing}_{\mathcal{G}} f = \{a_1, \ldots, a_k\}$. Let $D \subset \mathbb{C}$ be a small enough closed disc centered at a and let $s \in \partial D$. Then, for any small enough balls $B_i \subset \mathcal{Z} \times \mathbb{C}$ centered at a_i, we have:*

(a) $H_*(F_D, F_s) \simeq \oplus_{i=1}^k \tilde{H}^{2n-1-*}(B_i \cap \mathbb{X}_s)$.

(b) $H_*(B_i \cap F_D, B_i \cap F_s) \simeq \tilde{H}^{2n-1-*}(B_i \cap \mathbb{X}_s)$, $\forall i \in \{1, \ldots, k\}$.

Proof (a). A general Lefschetz duality result (see e.g. [Br, Prop. 5.2]) says that, since we work with triangulable spaces, we have:

$$H_*(F_D, F_s) \simeq H^{2n-*}(\mathbb{X}_D, \mathbb{X}_s).$$

Next, the cohomology group splits, through excision, into local contributions, by our localization result Prop. 3.6:

$$H^*(\mathbb{X}_D, \mathbb{X}_s) = \oplus_{i=1}^k H^*(B_i \cap \mathbb{X}_D, B_i \cap \mathbb{X}_s) = \oplus_{i=1}^k \tilde{H}^{*-1}(B_i \cap \mathbb{X}_s),$$

where the second equality holds because $B_i \cap \mathbb{X}_s$ is contractible, for small enough ball B_i.

(b). The same Lefschetz duality result may be applied locally to yield:

$$H_*(B_i \cap F_D, B_i \cap F_s) \simeq H^{2n-*}(B_i \cap \mathbb{X}_D, B_i \cap \mathbb{X}_s).$$

Note that the decomposition $H_*(F_D, F_s) \simeq \oplus_{i=1}^k H_*(B_i \cap F_D, B_i \cap F_s)$ also follows from the localization (Proposition 3.6). $\square$

For the relative homotopy type of (X, F) and (F_D, F) in case of isolated singularities, we prove the following relative bouquet theorem.

3.8 Theorem
Let f have isolated singularities with respect to some $\partial\tau$-stratification $\mathcal{G}$ relative to $\mathbb{X}^{\mathrm{pol}}$. Let F be a general fibre of f and $D \subset \mathbb{C}$ be an open disc. Then the space X, resp. F_D, is obtained from F to which one attaches a number of cells of real dimension $n = \dim \mathcal{Z}$. In particular we have the following homotopy equivalences:

(a) $X/F \simeq \bigvee S^n$.

(b) $F_D/F \simeq \bigvee S^n$.

The number of spheres will be discussed in the next section (in particular, Corollary 4.10). Theorem 3.8 extends the results of [ST-1, Theorem 3.1] and [Ti-1, Theorem 4.6] about polynomial functions on affine complex manifolds. We shall give the proof in §4., after introducing and proving a few technical ingredients.

When specializing to a Stein, highly connected space X, we obtain the following bouquet result:

3.9 Corollary *If the space $\mathcal{Z} \setminus \mathrm{Pol}(f)$ is Stein and $(n-1)$-connected, then the general fibre of f has the homotopy type of a bouquet of spheres $\bigvee S^{n-1}$.* $\square$

This extends the bouquet result of the authors [ST-1] in case of a polynomial function $g : \mathbb{C}^n \to \mathbb{C}$, since we may take $f = \tilde{g}/x_0^d$, as explained in §2. In this case, the space $\mathcal{Z} \setminus \mathrm{Pol}(f)$ is $\mathbb{C}^n$.

4 Polar loci and Milnor numbers at $\mathrm{Pol}(f)$

We show first that an isolated $\mathcal{G}$-singularity at a point of $\mathbb{X}^{\mathrm{pol}}$ is detectable by the presence of a certain local polar locus, which we define as follows.

4.1 Definition Let $\xi = (x, a) \in \mathbb{X}^{\mathrm{pol}} \setminus \mathbb{X}_\infty$ and consider a small neighbourhood $V \subset \mathcal{Z}$ of x, where $f_x = p/q$. Let $\operatorname{Sing}\pi$, respectively $\operatorname{Sing}(\pi, q)$, denote the singular locus of the map $\pi : X \cap V \times \mathbb{C} \to \mathbb{C}$, resp. $(\pi, q) : X \cap V \times \mathbb{C} \to \mathbb{C}^2$.

The *polar locus* $\Gamma_\xi(\pi, q)$ is the germ at ξ of the space:

$$\operatorname{closure}\{\operatorname{Sing}(\pi, q) \setminus (\operatorname{Sing}\pi \cup \mathbb{X}^{\mathrm{pol}})\} \subset \mathbb{X}.$$

Since X is isomorphic to $\mathcal{Z} \setminus \operatorname{Pol}(f)$, we also get the isomorphisms:

$$\Gamma_\xi(\pi, q) \simeq \Gamma_\xi(f, q) \simeq \Gamma_\xi(p, q).$$

The polar locus depends on the multiplicative unit u, i.e. $\Gamma_\xi(\pi, qu)$ is different from $\Gamma_\xi(\pi, q)$. Nevertheless, the polar locus may induce well defined invariants, as we show in the following.

4.2 Proposition *Let $\xi = (x, a) \in \mathbb{X}^{\mathrm{pol}} \setminus \mathbb{X}_\infty$ and let $f_x = p/q$. Let f have an isolated $\mathcal{G}$-singularity at ξ. Then:*

(a) *For any multiplicative unit u, the polar locus $\Gamma_\xi(\pi, qu)$ is either void or* $\dim \Gamma_\xi(\pi, qu) = 1$.

(b) *The intersection multiplicity* $\operatorname{mult}_\xi(\Gamma_\xi(\pi, qu), \mathbb{X}_a)$ *is independent on the unit u.*

Proof (a). Let $\mathbb{P}T^*_q = \mathbb{P}T^*_{q|\mathbb{X}\cap V\times\mathbb{C}}$ denote the projectivised relative conormal of q. The key argument we shall use here is the independence of $\mathbb{P}T^*_{qu}$ from the multiplicative unit u, which was proved by Tibăr [Ti-1, Prop. 3.2].

Since $\mathbb{P}T^*(V \times \mathbb{C})$ can be identified with $V \times \mathbb{C} \times \check{\mathbb{P}}^n$, where $\check{\mathbb{P}}^n$ denotes the space of hyperplanes through 0 in $\mathbb{C}^{n+1}$, we may consider the projections $\mathrm{pr}_1 : \mathbb{P}T^*_q \to \mathbb{X} \cap V \times \mathbb{C}$ and $\mathrm{pr}_2 : \mathbb{P}T^*_q \to \check{\mathbb{P}}^n$. Then $\Gamma_\xi(\pi, q)$ is the germ of $\mathrm{pr}_1(\mathrm{pr}_2^{-1}\{\bar{\pi} = a\})$ at ξ, where $\{\bar{\pi} = a\}$ denotes the hyperplane $\mathcal{Z} \times \{a\} \subset \mathcal{Z} \times \mathbb{C}$ and is identified to a point of $\check{\mathbb{P}}^n$.

Now $\dim \mathbb{P}T^*_q = n + 1$ and therefore $\Gamma_\xi(\pi, q)$ is either void or of dimension at least 1. On the other hand, by the (*) condition we get $\mathbb{X}_a \cap \operatorname{Sing}_{\mathcal{G}} f \supset \mathbb{X}_a \cap \mathrm{pr}_1(\mathrm{pr}_2^{-1}\{\bar{\pi} = a\})$ and since ξ is an isolated point, it follows that $\Gamma_\xi(\pi, q)$ has dimension at most 1.

More precisely, the polar locus at ξ is not void (hence a curve) if and only if $\mathrm{pr}_1(\mathrm{pr}_2^{-1}\{\bar{\pi} = a\}) \cap \mathbb{X}^{\mathrm{pol}} = \{\xi\}$ and this, if and only if $(\xi, \{\bar{\pi} = a\}) \in \mathbb{P}T^*_q$. Since $\mathbb{P}T^*_q = \mathbb{P}T^*_{qu}$, this last condition does not depend on u. Our claim is proved.

(b). Suppose $\Gamma_\xi(\pi, q)$ has dimension 1 (since if void, the multiplicity in cause is zero). Consider a small enough ball $B \subset \mathcal{Z} \times \mathbb{C}$ centered at ξ, to fit in the Milnor-Lê fibration [Le-2] of the function $\bar{\pi}$ at ξ:

(7) $$\bar{\pi}_| : B \cap \mathbb{X}_{D^*} \to D^*,$$

where $D \subset \mathbb{C}$ is centered at a. The notation $\Gamma(\pi, q)$ will stay for the representative in B of the germ $\Gamma_\xi(\pi, q)$. We may choose D so small that, for all $s \in \partial D$, those intersection points $\mathbb{X}_s \cap \Gamma(\pi, q)$ which tend to ξ when $s \to a$, are inside B. This is possible because $\Gamma(\pi, q)$ is a curve which cuts $\mathbb{X}^{\mathrm{pol}}$ at ξ.

We shall compute the homology $H_*(B \cap \mathbb{X}_s)$ of the Milnor fibre of the fibration (7). Inside B, the restriction of the function q to $B \cap \mathbb{X}_s$ has a finite number of isolated singularities, which are precisely the points of intersection $B \cap \mathbb{X}_s \cap \Gamma(\pi, q)$.

We start with the claim that the space $B \cap \mathbb{X}_s \cap q^{-1}(\hat{\delta})$ is contractible, for small enough disc $\hat{\delta} \subset \mathbb{C}$ centered at 0. We need the following:

4.3 Lemma *Let f have isolated $\mathcal{G}$-singularities at ξ. Let B be a small enough ball at ξ such that the sphere $S := \partial \bar{B}$ cuts transversely all those finitely many strata of $\mathcal{G}$ which have ξ in their closure and does not intersect other strata.*

Then, there exist small enough discs D and δ such that $(\bar{\pi}, q)^{-1}(\nu)$ is transverse to S, for all $\nu \in D \times \delta^*$.

Proof By absurd, if the statement is not true, then there exists a sequence of points $\eta_i \in S \cap (\mathbb{X} \setminus \mathbb{X}^{\mathrm{pol}})$ tending to a point $\eta \in S \cap \mathbb{X}_a \cap \mathbb{X}^{\mathrm{pol}}$, such that the intersection of tangent spaces $T_{\eta_i} \pi^{-1}(\pi(\eta_i)) \cap T_{\eta_i} q^{-1}(q(\eta_i))$ is contained in $T_{\eta_i}(S \cap X)$. Assuming, without loss of generality, that the following limits exist, we get:

$$\lim T_{\eta_i} \pi^{-1}(\pi(\eta_i)) \cap \lim T_{\eta_i} q^{-1}(q(\eta_i)) \subset \lim T_{\eta_i}(S \cap X). \tag{8}$$

Let $\mathcal{G}_\alpha \subset \mathbb{X}^{\mathrm{pol}}$ be the stratum containing η. Remark that $\dim \mathcal{G}_\alpha \geq 2$, since $\bar{\mathcal{G}}_\alpha \ni \xi$ and $\bar{\pi} \pitchfork_\eta \mathcal{G}_\alpha$. This implies that $\dim \mathcal{G}_\alpha \cap \mathbb{X}_a \geq 1$.

We have, by the definition of the stratification $\mathcal{G}$, that $\lim T_{\eta_i} q^{-1}(q(\eta_i)) \supset T_\eta \mathcal{G}_\alpha$ and obviously $T_\eta(\mathcal{G}_\alpha \cap \mathbb{X}_a) \subset T_\eta \mathcal{G}_\alpha$. On the other hand, $\lim T_{\eta_i} \pi^{-1}(\pi(\eta_i)) \supset T_\eta(\mathcal{G}_\alpha \cap \mathbb{X}_a)$, since $\bar{\pi} \pitchfork_\eta \mathcal{G}_\alpha$. In conclusion, the intersection in (8) contains $T_\eta(\mathcal{G}_\alpha \cap \mathbb{X}_a)$. But, since $S \pitchfork_\eta \mathcal{G}_\alpha$, the limit $\lim T_{\eta_i}(S \cap X)$ cannot contain $T_\eta(\mathcal{G}_\alpha \cap \mathbb{X}_a)$ and this gives a contradiction. $\square$

Let $\hat{\delta}$ be so small that $B \cap \mathbb{X}_s \cap q^{-1}(\hat{\delta}) \cap \Gamma(\pi, q) = \emptyset$. By the Lemma 4.3 above and by choosing apropriate D and $\hat{\delta}$, the map $q : B \cap \mathbb{X}_s \cap q^{-1}(\hat{\delta}^*) \to \hat{\delta}^*$ is a locally trivial fibration. Therefore $B \cap \mathbb{X}_s \cap q^{-1}(\hat{\delta})$ is homotopy equivalent, by retraction, to the central fibre $B \cap \mathbb{X}_s \cap \mathbb{X}^{\mathrm{pol}}$. This proves our claim.

We now remark that the central fibre $B \cap \mathbb{X}_s \cap \mathbb{X}^{\mathrm{pol}}$ is just the complex link at ξ of the space $\mathbb{X}^{\mathrm{pol}}$. The space $\mathbb{X}^{\mathrm{pol}}$ is a product $(\mathrm{Pol}(f) \cap \mathrm{Z}(f)) \times \mathbb{C}$ at ξ, along the projection axis $\mathbb{C}$, hence its complex link is contractible. Hence so is $B \cap \mathbb{X}_s \cap q^{-1}(\hat{\delta})$.

Pursuing the proof of Proposition 4.2, we observe that $B \cap \mathbb{X}_s$ is homotopy equivalent[1] to $B \cap \mathbb{X}_s \cap q^{-1}(\delta)$, for D and δ like in Lemma 4.3 and, in addition, the radius of D much smaller than the radius of δ. This supplementary condition is meant to insure that $\Gamma(\pi, q) \cap B \cap \mathbb{X}_s = \Gamma(\pi, q) \cap B \cap \mathbb{X}_s \cap q^{-1}(\delta)$.

Now, the total space $B \cap \mathbb{X}_s \cap q^{-1}(\delta)$ is built by attaching to the space $B \cap \mathbb{X}_s \cap q^{-1}(\hat{\delta})$, which is contractible, a finite number of cells of dimension $n-1$, which correspond to the Milnor numbers of the isolated singularities of the function q on $B \cap \mathbb{X}_s \cap q^{-1}(\delta \setminus \delta^*)$. The sum of these numbers is, by definition, the intersection multiplicity $\mathrm{mult}_\xi(\Gamma(\pi, q), \mathbb{X}_a)$.

We have proven that:

$$\dim H_{n-1}(B \cap \mathbb{X}_s) = \mathrm{mult}(\Gamma(\pi, q), \mathbb{X}_a) \text{ and } \tilde{H}_i(B \cap \mathbb{X}_s) = 0, \text{ for } i \neq n-1. \tag{9}$$

[1] exercice with the definition of homotopy equivalence

When replacing all over in our proof the function q by qu, we get the same relation (9), with qu instead of q. This concludes our proof of 4.2. $\square$

The above proof shows that $B \cap \mathbb{X}_s$ is, homotopically, a ball to which one attaches a certain number of $(n-1)$-cells. Thus, we get the following corollary and definition.

4.4 Corollary *Let f have an isolated $\mathcal{G}$-singularity at ξ. The fibre $B \cap \mathbb{X}_s$ of the local fibration (7) is homotopy equivalent to a bouquet of spheres $\bigvee S^{n-1}$ of dimension $n-1$.* $\square$

4.5 Definition We denote the number of spheres by $\lambda_\xi := \dim H_{n-1}(B \cap \mathbb{X}_s)$ and call it the *polar Milnor number* at ξ.

If f has isolated $\mathcal{G}$-singularities at $a \in \mathbb{C}$, we denote by λ_a the sum of the polar Milnor numbers at singularities on $\mathbb{X}_a \cap \mathbb{X}^{\rm pol}$. Also denote $\lambda = \sum_{a \in \Lambda_f} \lambda_a$.

From Proposition 3.7, we now get:

4.6 Corollary *If f has isolated $\mathcal{G}$-singularity at $a \in \mathbb{C}$ then the vanishing homology $H_*(F_D, F_s)$ is concentrated in dimension n.* $\square$

4.7 Definition Let f have an isolated $\mathcal{G}$-singularity at $\xi \in \mathbb{X}_a \cap \mathbb{X}^{\rm pol}$. We say that f *has vanishing cycles at ξ* if $\lambda_\xi > 0$.

We give in the following the proof of a previously stated result.

4.8 Proof of Theorem 3.8

We take back the notations of Theorem 3.8. Since $\mathrm{Sing}_{\mathcal{G}} f$ is a finite set of points on $\mathbb{X} \setminus \mathbb{X}_\infty$, the variation of topology of the fibres of f is localisable at those points (cf. Proposition 3.6). Let $\mathbb{X}_a \cap \mathrm{Sing}_{\mathcal{G}} f = \{a_1, \ldots, a_k\}$.

For a point $a_i \in X \cap \mathrm{Sing}_{\mathcal{G}} f$, it follows by the classical result of Milnor for holomorphic functions with isolated singularity [Mi] that the pair $(B_{\varepsilon,i} \cap X \cap F_{D_a}, B_{\varepsilon,i} \cap X \cap F_\eta)$ is $(\dim \mathcal{Z} - 1)$-connected, where $\eta \in D_a^*$.

For $a_i \in \mathbb{X}^{\rm pol} \cap \mathrm{Sing}_{\mathcal{G}} f$, a similar statement turns out to be true. We may invoke the following lemma, which is an extended version of a result by Hamm and Lê [HL-2, Corollary 4.2.2]:

Lemma ([Ti-1, Cor. 2.7])
The pair $(B_{\varepsilon,i} \cap F_{D_a}, B_{\varepsilon,i} \cap F_\eta)$ is $(n-1)$-connected, where $\eta \in D_a^*$.

We conclude that the space X is built up starting from a fibre F, then moving it within a fibration with a finite number of isolated singularities. By the above connectivity results and by Switzer's result [Sw, Proposition 6.13], at each singular point one has to attach a number of n-cells, equal to the local Milnor number, resp. the polar Milnor number. The total number of cells is the sum of all these Milnor numbers. $\square$

4.9 Corollary *If f has an isolated $\mathcal{G}$-singularity at ξ the we have the homotopy equivalence $B_{\varepsilon,i} \cap F_{D_a}/B_{\varepsilon,i} \cap F_\eta \simeq \bigvee S^n$.* □

As another consequence, we get the Betti numbers of the vanishing homology in case of isolated $\mathcal{G}$-singularities (see Proposition 3.7). This result extends the known formula in case of polynomial functions [ST-1, Corollary 3.5].

4.10 Corollary *Let f have isolated $\mathcal{G}$-singularities at $a \in \mathbb{C}$ with respect to some $\partial\tau$ stratification $\mathcal{G}$. Then:*

$$b_n(F_D, F_s) = (-1)^n \chi(F_D, F_s) = \mu_a + \lambda_a \ , \qquad H_j(F_D, F_s) = 0, \ \text{for } j \neq n,$$

where μ_a is the sum of the Milnor numbers of the singularities of F_a and λ_a denotes the sum of the polar Milnor numbers at $\mathbb{X}_a \cap \mathbb{X}^{\mathrm{pol}}$.

In particular, if f has isolated $\mathcal{G}$-singularities at all fibres, then:

$$b_n(X, F) = (-1)^n \chi(X, F) = \mu + \lambda \ , \qquad H_j(X, F) = 0, \ \text{for } j \neq n,$$

where μ is the total Milnor number of the singularities of f on $\mathcal{Z} \setminus \mathrm{Pol}(f)$ and λ is the total polar Milnor number at $\mathbb{X}^{\mathrm{pol}} \setminus \mathbb{X}_\infty$. □

4.11 Equisingularity along $\mathrm{Pol}(f)$**.**

Definition We say that f is C^∞-trivial along $\mathrm{Pol}(f)$ at $\xi \in \mathbb{X}^{\mathrm{pol}} \setminus \mathbb{X}_\infty$, resp. at $a \in \mathbb{C}$, if there is a neighbourhood $\mathcal{N}$ of ξ, resp. of $\mathbb{X}_a \cap \mathbb{X}^{\mathrm{pol}}$, and a small enough disc D at $a \in \mathbb{C}$ such that the map $\pi_| : \mathcal{N} \cap \pi^{-1}(D) \to D$ is a C^∞-trivial fibration.

As in [Ti-3], the consideration of the $\partial\tau$ condition (*) leads to an equisingularity condition along the divisor $\mathrm{Pol}(f)$, which we call *π-equisingularity along* $\mathrm{Pol}(f)$. Following the arguments in [Ti-3], C^∞-triviality along $\mathrm{Pol}(f)$ is implied by π-equisingularity along $\mathrm{Pol}(f)$, cf. [Ti-3, Theorem 2.7]. Next, from [Ti-3, Theorem 4.6] and the remark following it, respectively [Ti-3, Theorem 1.2], we deduce the following results:

4.12 Theorem *Let f have isolated $\mathcal{G}$-singularities at ξ, resp. at $a \in \mathbb{C}$. Then f is C^∞-trivial along $\mathrm{Pol}(f)$ at ξ, resp. at $a \in \mathbb{C}$ if and only if $\lambda_\xi = 0$, resp. $\lambda_a = 0$.*

Moreover, if f has isolated $\mathcal{G}$-singularities at $a \in \mathbb{C}$, then F_a is a general fibre of f if and only if $\lambda_a = 0$ and $\mu_a = 0$. □

Let us notice that, in the notations above, we have $\chi(F_D) = \chi(F_a)$ and therefore $\chi(F_D, F_s) = \chi(F_a) - \chi(F_s)$. Then, combining Theorem 4.12 with Corollary 4.10, we get the following consequence, which generalises the criteria for atypical fibres in case of polynomial functions in 2 variables [HaLe] and in n variables [ST-1], [Pa].

4.13 Corollary *Let f have isolated $\mathcal{G}$-singularities at $a \in \mathbb{C}$. Then F_a is a typical fibre if and only if $\chi(F_a) = \chi(F)$, where F is a general fibre of f.* □

5 Vanishing cycles in special cases and examples

The singular locus $\operatorname{Sing}\mathbb{X} \subset \mathcal{Z}\times\mathbb{C}$ is contained in $\mathbb{X}^{\rm pol}$ and can be complicated. We have $\operatorname{Sing}\mathbb{X}\setminus\mathbb{X}_\infty = \cup_{t\in\mathbb{C}}(\operatorname{Sing}\mathbb{X}_t)\cap\mathbb{X}^{\rm pol}$. However, $\mathbb{X}^{\rm pol}\setminus\operatorname{Sing}\mathbb{X}$ is a Whitney stratum and $\operatorname{Sing}\mathbb{X}$ is a union of Whitney strata, in the canonical Whitney stratification $\mathcal{W}$ of $\mathbb{X}$ which has X as a stratum.

We shall consider here a $\partial\tau$-stratification $\mathcal{G}$ which is coarser than $\mathcal{W}$ (which exists, by Definition 3.3 and the remark following it). Then $\operatorname{Sing}_{\mathcal{G}}f\cap\mathbb{X}^{\rm pol}\subset\operatorname{Sing}\mathbb{X}$. Indeed, this follows from the fact that the space $\mathbb{X}^{\rm pol}\setminus\mathbb{X}_\infty$ is locally a product $\{q=p=0\}\times\mathbb{C}$ and the projection $\bar\pi$ is transversal to it off $\operatorname{Sing}\mathbb{X}$.

In particular, for $n=2$, f has isolated $\mathcal{G}$-singularities at a if and only if F_a is reduced.

Let $\xi=(x,a)\in\mathbb{X}^{\rm pol}\setminus\mathbb{X}_\infty$. We assume in the following that $\dim_\xi\operatorname{Sing}\mathbb{X}_a=0$. This implies that $\dim_\xi\operatorname{Sing}_{\mathcal{G}}f\le 0$ and that the germ $(\operatorname{Sing}\mathbb{X},\xi)$ is either a curve or just the point ξ. If a curve, then it can have several branches and its intersection with $\mathbb{X}_s$ is, say, $\{\xi_1(s),\ldots,\xi_k(s)\}$, for any $s\in D^*$, where $D\subset\mathbb{C}$ is a small enough disc at a.

The germs $(\mathbb{X}_s,\xi_i(s))$ are germs of hypersurfaces with isolated singularity. Let $\mu_i(s)$ denote the Milnor number of $(\mathbb{X}_s,\xi_i(s))$. Then $\sum_{i=1}^k\mu_i(s)\le\mu(a)$. Equality may hold only if $k=1$, by the well known nonsplitting result of Lê D.T [Le-1]. In general, we have:

5.1 Theorem *Let* $\dim_\xi\operatorname{Sing}\mathbb{X}_a=0$ *and* $\dim_\xi\operatorname{Sing}\mathbb{X}=1$. *Then:*

$$\lambda_\xi=\mu(a)-\sum_{i=1}^k\mu_i(s).$$

In particular, there are vanishing cycles at ξ *if and only if* $\lambda_\xi>0$.

Proof The hypothesis implies that the germ of $\operatorname{Sing}_{\mathcal{G}}f$ at ξ is just the point ξ. For any $s\in D$ small enough, the germ $(\mathbb{X}_s,\xi_i(s))$ is locally defined by the function:

$$F=p-tq:(\mathcal{Z}\times\mathbb{C},\xi_i(s))\to\mathbb{C}.$$

We have that, locally at ξ, the singular locus $\operatorname{Sing}F$ is equal to $\operatorname{Sing}\mathbb{X}$, in particular included into $\mathbb{X}^{\rm pol}$. Consider the map $(F,t):(\mathcal{Z}\times\mathbb{C},\xi_i(s))\to\mathbb{C}^2$. Note that the polar locus $\Gamma_\xi(F,t)$ is a curve or it is void, since ξ is an isolated $\mathcal{G}$-singularity. Following [Le-2], see also [Ti-2], there is a fundamental system of privileged polydisc neighbourhoods of ξ in $\mathcal{Z}\times\mathbb{C}$, of the form $(P_\alpha\times D'_\alpha)$, where $D'_\alpha\subset\mathbb{C}$ is a disc at a and P_α is a polydisc at $x\in\mathcal{Z}$ such that the map

$$(F,t):(\mathcal{Z}\times\mathbb{C})\cap(P_\alpha\times D'_\alpha)\cap(F,t)^{-1}(D_\alpha\times D'_\alpha)\to D_\alpha\times D'_\alpha$$

is a locally trivial fibration over $(D^*_\alpha\times D'_\alpha)\setminus\operatorname{Im}(\Gamma(F,t))$. We chose D_α and D'_α such that $\operatorname{Im}(\Gamma(F,t))\cap\partial(\overline{D^*_\alpha\times D'_\alpha})=\operatorname{Im}(\Gamma(F,t))\cap(D^*_\alpha\times\partial(\overline{D'_\alpha}))$. Let $s\in\partial\overline{D'_\alpha}$. Observe that $t^{-1}(s)\cap(P_\alpha\times D'_\alpha)$ is contractible, since it is the Milnor fibre of the linear function t on a smooth space. This is obtained, up to homotopy type, by attaching to $(F,t)^{-1}(0,s)\cap$

$(P_\alpha \times D'_\alpha)$ a certain number r of n-cells, equal to the sum of the Milnor numbers of the function $F_| : t^{-1}(s) \cap (P_\alpha \times D'_\alpha) \to D_\alpha$. Since we have the homotopy equivalence $(F,t)^{-1}(0,s) \cap (P_\alpha \times D'_\alpha) \simeq B \cap \mathbb{X}_s$, we get, by Corollary 4.4 and Definition 4.5, that $r = \lambda_\xi$.

Now $(F,t)^{-1}(\eta,s) \cap (P_\alpha \times D'_\alpha)$ is homotopy equivalent to the Milnor fibre of the germ $(\mathbb{X}_a, \xi)$, which has Milnor number $\mu(a)$. The space $t^{-1}(s) \cap (P_\alpha \times D'_\alpha)$ is obtained from $(F,t)^{-1}(\eta,s) \cap (P_\alpha \times D'_\alpha)$ by attaching exactly r cells of dimension n (comming from the polar intersections) and of a number of n-cells comming from the intersections with $\operatorname{Sing} F$. This number of cells is, by definition, $\sum_{i=1}^k \mu_i(s)$. We get the equality:

$$\mu(a) = r + \sum_{i=1}^{k} \mu_i(s).$$

Lastly, since $r = \lambda_\xi$, our proof is done. $\square$

5.2 Remark If in the hypothesis of Theorem 5.1 the dimension of $\operatorname{Sing} \mathbb{X}$ is not 1 but 0, then the result still holds, with the remark that in this case $\mu_i(s) = 0$, $\forall i$ and $\forall s \in D^*$. Hence $\lambda_\xi = \mu(a)$.

We give in the remainder two examples.

5.3 Example $E^{a,b}_{p,q}$: $\quad f = \dfrac{x(z^{a+b} + x^a y^b)}{y^p z^q}$, with $a+b+1 = p+q$ and $a,b,p,q \geq 1$. This defines a meromorphic function on $\mathbb{P}^2(\mathbb{C})$. For some $t \in \mathbb{C}$, the space $\mathbb{X}_t$ is given by:

$$x(z^{a+b} + x^a y^b) = t y^p z^q \tag{10}$$

We have $\mathbb{X}^{\mathrm{pol}} \setminus \mathbb{X}_\infty = \{[1:0:0], [0:1:0], [0:0:1]\} \times \mathbb{C}$. According to Theorem 5.1, we look for jumps in the Milnor number within the family of germs (10):

(a) at $[1:0:0]$, chart $x = 1$. No jumps, since uniform Brieskorn type $(b, a+b)$.

(b) at $[0:1:0]$, chart $y = 1$. For $t \neq 0$, Brieskorn type $(a+1, q)$, with $\mu(t) = a(q-1)$. If $t = 0$, then we have $x^{a+1} + xz^{a+b} = 0$ with $\mu(0) = a^2 + ab + b$ and the jump at $\xi = ([0:1:0], 0)$ is $\lambda_\xi = a^2 + ab + b - a(q-1) = b + ap$, by Theorem 5.1.

(c) at $[0:0:1]$, chart $z = 1$. No jumps, since type A_0 for all t.

We get the total jump $\lambda = b + ap$. A straightforward computation shows that $\mu = 0$.

The fibres of f can be described as follows. If $t = 0$, we have $c+1$ disjoint copies of $\mathbb{C}^*$, where $c = \gcd(a,b)$; hence $\chi(F_0) = 0$. If $t \neq 0$, we compute $\chi(F) = -(b+ap)$, by a branched covering argument. The vanishing homology is concentrated in dimension 2. Taking $X = \mathbb{C}^2 \setminus \{y = 0\}$, we get the Betti number $b_2(X,F) = \chi(X,F) = \chi(X) - \chi(F) = 0 + (b+ap) = b + ap$. It follows $b_2(X,F) = \lambda + \mu$, which agrees with Corollary 4.10.

5.4 Example Consider the meromorphic function $f = \dfrac{x(z^2+xy)}{z^3}$ on the smooth hypersurface $\mathcal{Z} \subset \mathbb{P}^3$ given by $h = yw + x^2 - z^2 = 0$. Then $\mathbb{X}^{\mathrm{pol}} \setminus \mathbb{X}_\infty = [0:0:0:1] \times \mathbb{C} \cup [0:1:0:0] \times \mathbb{C}$, where $[x:y:z:w]$ are the homogeneous coordinates in $\mathbb{P}^3$.

Along $[0:0:0:1] \times \mathbb{C}$, in the chart $w = 1$ and coordinates x and z on $\mathbb{X}$, we have the family of curves (germs of $\mathbb{X}_t$):

$$x(z^2 - x^3 + xz^2) = tz^3. \tag{11}$$

For all t, this is a D_5 singularity, so no jumps.

Along $[0:1:0:0] \times \mathbb{C}$, in the chart $y = 1$ and, again, x and z as coordinates on $\mathbb{X}$, we have the family of curves (germs of $\mathbb{X}_t$):

$$x(z^2 + x) = tz^3. \tag{12}$$

This has type A_2 if $t \neq 0$ and A_3 if $t = 0$. Thus the jump at $\xi := ([0:1:0:0], 0)$ is $\lambda_\xi = 1$ and the total jump is $\lambda = 1$.

By simple computations, we get $\mu = 2$, since there are two singular fibres, $F_{\pm 1}$, with A_1-singularities. There are 3 atypical fibres: $F_0 \simeq \mathbb{C}^* \sqcup \mathbb{C}^*$, $F_{\pm 1} \simeq \mathbb{C}^*$ and the general fibre $F \simeq \mathbb{C}^{**}$. Since $X \simeq S^2$, we get $b_2(X, F) = 2 - (-1) = 3$ global vanishing cycles, $X/F \simeq \bigvee_3 S^2$.

References

[A'C] N. A'Campo, *Le nombre de Lefschetz d'une monodromie*, Indag. Math. **35** (1973), 113–118.

[Ar] V.I. Arnold, *Singularities of fractions and behaviour of polynomials at infinity*, preprint ESN 1998.

[ACD] E. Artal-Bartolo, P. Cassou-Nogués, A. Dimca, *Sur la topologie des polynomes complexes*, in: eds. Arnold, V. I. et al., "Singularities. The Brieskorn Anniversary Volume", Prog. in Math. 162, Birkhäuser 1998, p. 317-343.

[BMM] J. Briançon, Ph. Maisonobe, M. Merle, *Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom* , Inventiones Math., **117**, 3 (1994), 531–550.

[Br] S.A. Broughton, *On the topology of polynomial hypersurfaces*, Proceedings A.M.S. Symp. in Pure. Math., vol. 40, I (1983), 165-178.

[GLM-1] S. Gusein-Zade, I. Luengo, A. Melle, *Zeta functions for germs of meromorphic functions and Newton diagrams*, Funct. Anal. Appl. (1998).

[GLM-2] S. Gusein-Zade, I. Luengo, A. Melle, *On atypical values and local monodromies of meromorphic functions*, ESN preprint server http://www.mi.aau.dk∼esn.

[GM] M. Goresky, R. MacPherson, *Stratified Morse Theory*, Springer-Verlag Berlin Heidelberg New-York, 1987.

[GWPL] C.G. Gibson, K. Wirthmüller, A.A. du Plessis, E.J.N. Looijenga, *Topological Stability of Smooth Mappings*, Lect. Notes in Math. **552**, Springer Verlag 1976.

[GR] R.C. Gunning, H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.

[HaLe] Hà H.V., Lê D.T., *Sur la topologie des polynômes complexes*, Acta Math. Vietnamica, **9** (1984), pp. 21–32.

[HL-1] H. Hamm, Lê D.T., *Un théorème de Zariski du type de Lefschetz*, Ann. Scient. Ec. Norm. Sup. $\mathbf{4}^e$ série, t.6 (1973), 317–366.

[HL-2] H. Hamm, Lê D.T., *Rectified homotopical depth and Grothendieck conjectures*, in: P. Cartier et all. (eds) Grothendieck Festschrift II, pp. 311–351, Birkhäuser 1991.

[HMS] J.P. Henry, M. Merle, C. Sabbah, *Sur la condition de Thom stricte pour un morphisme analytique complexe*, Ann. Scient. Ec. Norm. Sup. $\mathbf{4}^e$ série, t. 17 (1984), 227–268.

[Le-1] Lê D.T., *Une application d'un théorème d'A'Campo à l'équisingularité*, Nederl. Akad. Wet., Proc., Ser. A **76**, 403-409 (1973).

[Le-2] Lê D.T., *Some remarks on the relative monodromy*, in: Real and Complex Singularities, Oslo 1976, Sijhoff en Norhoff, Alphen a.d. Rijn 1977, p. 397–403.

[Mi] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies **61**, Princeton 1968.

[NN] W. D. Neumann, P. Norbury, *Vanishing cycles and monodromy of complex polynomials*, manuscript available at http://neumann.maths.mu.oz.au/preprints.html.

[Pa] A. Parusinski, *On the bifurcation set of a complex polynomial with isolated singularities at infinity*, Compositio Math. **97**, 3(1995), 369–384.

[ST-1] D. Siersma, M. Tibăr, *Singularities at infinity and their vanishing cycles*, Duke Math. Journal **80**:3 (1995), 771-783.

[ST-2] D. Siersma, M. Tibăr, *Singularities at infinity and their vanishing cycles, II. Monodromy*, preprint nr. 1088, january 1999, Universiteit Utrecht.

[Sw] R. Switzer, *Algebraic Topology - Homotopy and Homology*, Springer Verlag, Berlin-Heidelberg-New York 1975.

[Te] B. Teissier, *Varietés polaires 2: Multiplicités polaires, sections planes et conditions de Whitney*, Géométrie Algèbrique à la Rabida, Springer L.N.M. **961**, pp. 314–491.

[T] R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. 75 (1969), 249-312.

[Ti-1] M. Tibăr, *Topology at infinity of polynomial maps and Thom regularity condition*, Compositio Math., 111, 1 (1998), 89-109.

[Ti-2] M. Tibăr, *Embedding Nonisolated Singularities into Isolated Singularities*, in: Singularities, The Brieskorn Anniversary Volume, eds. V.I. Arnold, G.-M. Greuel, J.H.M. Steenbrink, Progress in Mathematics vol. 162, Birkhäuser Verlag 1998.

[Ti-3] M. Tibăr, *Asymptotic Equisingularity and Topology of Complex Hypersurfaces*, Int. Math. Research Notices, **18** (1998), 979-990.

[Ve] J.-L. Verdier, *Stratifications de Whitney et théorème de Bertini-Sard*, Inventiones Math. 36 (1976), 295-312.

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